An attempt to derive the conversion of variance to reliability

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When estimating the correlation between two variables, x and y, it is well known that if the measurement of either variable contains error, this error tends to lead to estimates that underestimate the true correlation, ρ_{xy}. It has been suggested that empirical estimates of measurement error can used to correct for this underestimation. Spearman (1904) suggested that ρ may be obtained by

$$\rho_{xy} = r_{xy} \times \gamma_{rel} \tag{1}$$

where r_{xy} is the correlation between x and y, and γ_{rel} is a correction factor defined as

$$\gamma_{rel} = \frac{1}{\sqrt{r_{xx} \times r_{yy}}} \tag{2}$$

where r_{xx} and r_{yy} are the reliabilities of x and y, respectively.

An alternative approach, developed by Liu et al (1978) suggests that observations of both within and between unit variance can be used to obtain ρ_{xy}:

$$\rho_{xy} = r_{xy} \times \gamma_{var} \tag{3}$$

where the correction factor, γ_{var}, is defined as

$$\gamma_{var} = \sqrt{(1 + \frac{\frac{\bar{\omega}_x}{\beta_x}}{\bar{k}_x}) \times (1 + \frac{\frac{\bar{\omega}_y}{\beta_y}}{\bar{k}_y})} \tag{4}$$

where $\bar{k}_x$ is the mean number of observations of x per unit of observation, β_x is the between unit variance observed in x and $\bar{\omega}_x$ is the mean within unit variance observed in x, with similar definitions of $\bar{k}_y$, β_y, and $\bar{\omega}_y$.

These parallel approaches to the correction of observed correlation to achieve estimates of true correlation also provide a means by which reliability may be estimated. That is, if Equations 1 & 3 hold, then it should be true that

$$\gamma_{rel} = \gamma_{var} \tag{5}$$

That is,

$$\frac{1}{\sqrt{r_{xx} \times r_{yy}}} = \sqrt{(1 + \frac{\frac{\bar{\omega}_x}{\beta_x}}{\bar{k}_x}) \times (1 + \frac{\frac{\bar{\omega}_y}{\beta_y}}{\bar{k}_y})} \tag{6}$$

Consider, then, the special case where the reliability of x and y is equivalent. This means that

$$\gamma_{rel} = \frac{1}{\sqrt{r_{xx} \times r_{yy}}} = \frac{1}{\sqrt{r_{xx}^2}} = \frac{1}{r_{xx}} \tag{7}$$

and

$$\gamma_{var} = \sqrt{(1 + \frac{\frac{\bar{\omega}_x}{\beta_x}}{\bar{k}_x})(1 + \frac{\frac{\bar{\omega}_y}{\beta_y}}{\bar{k}_y})} = \sqrt{(1 + \frac{\frac{\bar{\omega}_x}{\beta_x}}{\bar{k}_x})^2} = 1 + \frac{\frac{\bar{\omega}_y}{\beta_y}}{\bar{k}_x} \tag{8}$$

Using Equation 6 again,

$$\frac{1}{r_{xx}} = 1 + \frac{\frac{\bar{\omega}_x}{\beta_x}}{\bar{k}_x} \tag{9}$$

Finally, solving for r_{xx} we find that

$$r_{xx} = \frac{1}{1 + \frac{\frac{\bar{\omega}_x}{\beta_x}}{k_x}} \tag{10}$$